PERGAMON
Mathematical and Computer Modelling 35 (2002) 1365–1370

MATHEMATICAL AND COMPUTER MODELLING

www.elsevier.com/locate/mcm

Riabouchinsky Flow with Partially Penetrable Obstacle

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(Received and accepted February 2002)

Abstract—An explicitly solvable Riabouchinsky model with a partially penetrable obstacle is introduced. This model applied to the estimation of the efficiency of free flow turbines allows us to take into account the pressure drop past the lamina.

Keywords—Cavitation flows, Riabouchinsky model, Kirchhoff method, Free boundary problems.

1. INTRODUCTION

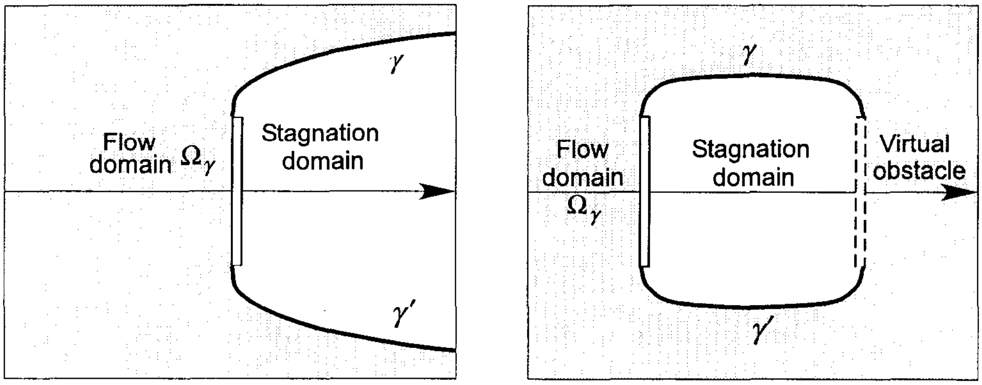
The investigation of hydrodynamic models with partially penetrable obstacles has been initiated by the recent progress in the development of free flow turbines [1] for the purpose of estimating their efficiency limit [2]. In the previous paper [3], an explicitly solvable analog of the Kirchhoff cavitation flow was obtained. As in the classical situation, the main deficiency of the Kirchhoff model is the infinite size of the stagnation domain that resulted from the assumption that the pressure in the stagnation domain is equal to the pressure at infinity. In the Riabouchinsky model (Figure 1b), a virtual obstacle past the actual one is introduced to make the stagnation domain finite. In this model, the separating streamlines γ and γ' join the edges of both obstacles. The velocity V_γ on γ and γ' is greater than the velocity at infinity V_∞ and the pressure in the stagnation domain is less than the pressure at infinity. The number σ, s.t.

$$\frac{V_\gamma^2}{V_\infty^2} = 1 + \sigma, \tag{1}$$

is called the cavitational number. For practical application, the Riabouchinsky model attains their importance because of its ability to deal with variably small cavitation numbers.

The authors are very grateful to Prof. A. M. Gorlov for helpful discussion.

0895-7177/02/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
PII: S0895-7177(02)00089-4



(a) Kirchhoff model. (b) Riabouchinsky model.

Figure 1. Kirchhoff and Riabouchinsky models.

2. MODIFIED RIABOUCHINSKY FLOW

The modified Riabouchinsky flow is constructed similarly to the modified Kirchhoff flow [2,3]. In the flow domain Ω_γ (Figure 1b), the flow is potential. Let us denote its potential w, i.e., a holomorphic function, s.t.

$$\vec{V} = \overline{\frac{dw}{dz}}, \tag{2}$$

where $\vec{V}$ is the velocity field. The Agrand diagram [4] of the modified Riabouchinsky flow is presented in Figure 2. (The current moves towards the positive direction of the x-axis.) In this situation, the flow is symmetric about the vertical line equally distant from both obstacles (with reverting the direction of the flow). Therefore, it suffices to consider only its left part. Since the efficiency we are interested in is dimensionless, assume for simplicity that the density of the fluid, the half of the length of the lamina, and the velocity at infinity are all equal to one. As in the case of modified Kirchhoff flow [2,3], the situation becomes explicitly solvable with the additional assumption that the flow crosses the lamina at the same angle α at any point. In this case, potential maps the left half of Ω_γ onto the shape shown in Figure 2b. The hodograph plane

$$\zeta = \xi + i\eta = \log \frac{dw}{dz} = \log V - i \arg \vec{V} \tag{3}$$

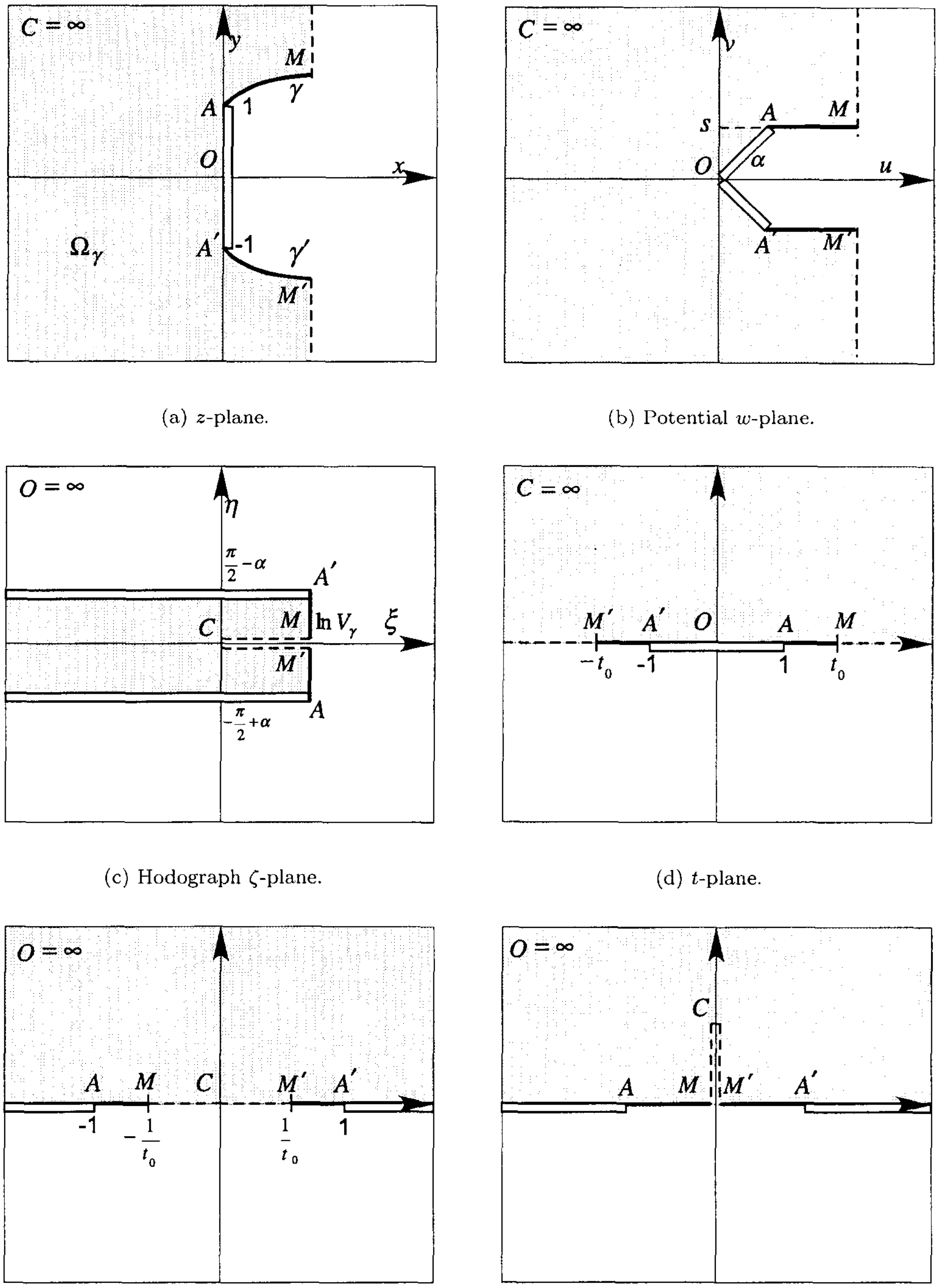
is shown in Figure 2c. Using the geometrical meaning of the hodograph given by (3), one can check that the conformal image the left half of Ω_γ is the semistrip having a cut along the positive part of the x-axis. The width of the semistrip is $2 - 4\alpha/\pi$, since the current crosses the lamina at the angle α.

As in the classical case and the case of modified Kirchhoff flow [2,3], the Kirchhoff method is still applicable in our situation. The essence of this method is to eliminate free boundaries γ and γ' from the problem by considering the potential and the hodograph planes, where they become straight. Since $\zeta = \log \frac{dw}{dz}$, the conformal representations of the shapes on w- and z-planes give the differential equation for w as the function of z that can be integrated.

Let the upper semiplane $\{t > 0\}$ be the canonical domain (Figure 2d). The conformal representation of the shape on the w-plane is constructed using the Christoffel-Schwarz integral

$$\frac{dw}{dt} = iK\left(t^2 - t_0^2\right)^{-1/2}\left(t^2 - 1\right)^{\alpha/\pi} t^{1-2\alpha/\pi}, \tag{4}$$

$$w = iK \int_0^1 \left(\tau^2 - t_0^2\right)^{-1/2}\left(\tau^2 - 1\right)^{\alpha/\pi} \tau^{1-2\alpha/\pi}\, d\tau, \tag{5}$$



(a) z-plane.

(b) Potential w-plane.

(c) Hodograph ζ-plane.

(d) t-plane.

(e) T-plane.

(f) a-plane.

Figure 2. Modified Riabouchinsky flow.

where t_0 and K are the constant to be determined. The distance s on the w-plane that can be interpreted as the distance between free streamlines at infinity or the fraction of the flow that passes through the obstacle can be computed from

$$\frac{se^{i\alpha}}{\sin\alpha} = iK \int_0^1 \left(\tau^2 - 1\right)^{\alpha/\pi} \left(\tau^2 - t_0^2\right)^{-1/2} \tau^{1-2\alpha/\pi}\, d\tau, \tag{6}$$

$$s = KI_1 \sin(\alpha), \tag{7}$$

where

$$I_1 = \int_0^1 \left(1-\tau^2\right)^{\alpha/\pi} \left(t_0^2-\tau^2\right)^{-1/2} \tau^{1-2\alpha/\pi}\, d\tau. \tag{8}$$

The conformal representation of semistrip with a cut on the ζ-plane with the boundary extension as shown in Figure 1c is constructed using the auxiliary a-plane and T-plane where

$$T = -\frac{1}{t} \tag{9}$$

and

$$a = \sqrt{\frac{T^2 - 1/t_0^2}{1-1/t_0^2}} = \frac{1}{t}\sqrt{\frac{t_0^2-t^2}{t_0^2-1}}. \tag{10}$$

Then

$$\sqrt{a^2-1} = \frac{t_0}{t}\sqrt{\frac{1-t^2}{t_0^2-1}} \tag{11}$$

and the conformal map we need is given by

$$\zeta = -\left(1-\frac{2\alpha}{\pi}\right)\ln\left(a+\sqrt{a^2-1}\right) - i\left(\frac{\pi}{2}-\alpha\right) + \ln V_\gamma \tag{12}$$

$$= -\left(1-\frac{2\alpha}{\pi}\right)\ln\left(\frac{1}{t}\sqrt{\frac{t_0^2-t^2}{t_0^2-1}} + \frac{t_0}{t}\sqrt{\frac{1-t^2}{t_0^2-1}}\right) - i\left(\frac{\pi}{2}-\alpha\right) + \ln V_\gamma. \tag{13}$$

The point $C = \infty$ maps to the origin, so

$$-\ln V_\gamma = -\left(1-\frac{2\alpha}{\pi}\right)\ln\left(\sqrt{\frac{-1}{t_0^2-1}} + t_0\sqrt{\frac{-1}{t_0^2-1}}\right) - i\left(\frac{\pi}{2}-\alpha\right), \tag{14}$$

$$\ln V_\gamma = \left(1-\frac{2\alpha}{\pi}\right)\ln\frac{1+t_0}{\sqrt{t_0^2-1}} = \left(1-\frac{2\alpha}{\pi}\right)\ln\sqrt{\frac{t_0+1}{t_0-1}}, \tag{15}$$

$$V_\gamma = \left(\frac{t_0+1}{t_0-1}\right)^{1/2-\alpha/\pi}. \tag{16}$$

Then

$$-\ln\frac{dz}{dw} = -\left(1-\frac{2\alpha}{\pi}\right)\ln\left(\frac{1}{t}\sqrt{\frac{t_0^2-t^2}{t_0^2-1}} + \frac{t_0}{t}\sqrt{\frac{1-t^2}{t_0^2-1}}\right) - i\left(\frac{\pi}{2}-\alpha\right) + \ln V_\gamma,$$

$$\frac{dz}{dw} = \frac{e^{i(\pi/2-\alpha)}}{V_\gamma}\left(\frac{1}{t}\sqrt{\frac{t_0^2-t^2}{t_0^2-1}} + \frac{t_0}{t}\sqrt{\frac{1-t^2}{t_0^2-1}}\right)^{1-2\alpha/\pi},$$

$$\frac{dz}{dt} = \frac{dz}{dw}\cdot\frac{dw}{dt} = \frac{iK}{V_\gamma}\left(\sqrt{\frac{t_0^2-t^2}{t_0^2-1}} + t_0\sqrt{\frac{1-t^2}{t_0^2-1}}\right)^{1-2\alpha/\pi} \left(1-t^2\right)^{\alpha/\pi}\left(t_0^2-t^2\right)^{-1/2}.$$

The constant K is determined from the relation $\int_0^1 \frac{dz}{i\,dt} = 1$

$$1 = \frac{K}{V_\gamma}\int_0^1 \left(\sqrt{\frac{t_0^2-t^2}{t_0^2-1}} + t_0\sqrt{\frac{1-t^2}{t_0^2-1}}\right)^{1-2\alpha/\pi} \left(1-t^2\right)^{\alpha/\pi}\left(t_0^2-t^2\right)^{-1/2} dt, \tag{17}$$

$$K = \frac{V_\gamma}{I_2}, \tag{18}$$

where

$$I_2 = \int_0^1 \left(\sqrt{\frac{t_0^2 - t^2}{t_0^2 - 1}} + t_0 \sqrt{\frac{1 - t^2}{t_0^2 - 1}} \right)^{1-2\alpha/\pi} \left(1 - t^2\right)^{\alpha/\pi} \left(t_0^2 - t^2\right)^{-1/2} dt. \tag{19}$$

The conformal representation of the left half of Ω_γ is given by

$$z(t) = \frac{K}{V_\gamma} \int_0^t \left(\sqrt{\frac{t_0^2 - \tau^2}{t_0^2 - 1}} + t_0 \sqrt{\frac{1 - \tau^2}{t_0^2 - 1}} \right)^{1-2\alpha/\pi} \left(1 - \tau^2\right)^{\alpha/\pi} \left(t_0^2 - \tau^2\right)^{-1/2} d\tau. \tag{20}$$

3. THE EFFICIENCY

The efficiency $\mathcal{E}$ of a free flow turbine was defined in [2,3] as the ratio of the power P absorbed by the obstacle to the power P_∞ carried by undisturbed flow through the projected area of the obstacle perpendicular to the flow. The power absorbed by the lamina is given by

$$P = \int_{-1}^1 [p] V_x \, dy = \frac{1}{2} \int_{-1}^1 V_x \left(V_\gamma^2 - V^2\right) dy, \tag{21}$$

where V_x is the x-component of the velocity $\vec{V}$ and $[p]$ denotes the pressure drop across the lamina which is equal to $(V_\gamma^2 - V^2)/2$ by the Bernoulli theorem. The power carried by the undisturbed flow through the lamina of width 2 is

$$P_\infty = 2 \cdot \frac{\rho V_\infty^3}{2} = 1, \tag{22}$$

since ρ and V_∞ are both equal to 1. Then

$$\mathcal{E} = \frac{P}{P_\infty} = \frac{1}{2} \int_{-1}^1 V_x \left(V_\gamma^2 - V^2\right) dy \tag{23}$$

$$= \int_0^1 V_x \left(V_\gamma^2 - V^2\right) dy \tag{24}$$

$$= \int_0^1 \left(\operatorname{Re} \frac{dw}{dz}\right) \left(V_\gamma^2 - V^2\right) dy \tag{25}$$

$$= \frac{1}{i} \int \left(\operatorname{Re} \frac{dw}{dz}\right) \left(V_\gamma^2 - V^2\right) \frac{dz}{dt} \, dt \tag{26}$$

$$= s V_\gamma^2 - \frac{1}{i} \int_0^1 \left(\operatorname{Re} \frac{dw}{dz}\right) \left|\frac{dw}{dz}\right|^2 \frac{dz}{i \, dt} \, dt \tag{27}$$

$$= s V_\gamma^2 - \sin\alpha \int_0^1 \left|\frac{dw}{dz}\right|^3 \frac{dz}{i \, dt} \, dt \tag{28}$$

$$= s V_\gamma^2 - \sin\alpha \frac{V_\gamma^3}{I_2} I_3 = \frac{V_\gamma^3}{I_2} \sin\alpha \left(I_1 - I_3\right), \tag{29}$$

where

$$I_3 = \int_0^1 \left(\sqrt{\frac{t_0^2 - t^2}{t_0^2 - 1}} + t_0 \sqrt{\frac{1 - t^2}{t_0^2 - 1}} \right)^{4\alpha/\pi - 2} \left(1 - t^2\right)^{\alpha/\pi} \left(t_0^2 - t^2\right)^{-1/2} t^{3-6\alpha/\pi} \, dt. \tag{30}$$

Table 1.

Inclination Angle, α	Cavitational Number, σ				
	0.02	0.04	0.06	0.08	0.1
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0785	0.0181	0.0186	0.0192	0.0197	0.0203
0.2356	0.0582	0.0599	0.0616	0.0634	0.0651
0.3927	0.1029	0.1060	0.1090	0.1121	0.1152
0.5497	0.1516	0.1560	0.1605	0.1651	0.1697
0.7068	0.2021	0.2081	0.2141	0.2202	0.2263
0.8639	0.2510	0.2584	0.2659	0.2735	0.2811
1.0210	0.2914	0.3000	0.3088	0.3176	0.3265
1.1781	0.3102	0.3195	0.3289	0.3385	0.3482
1.3351	0.2811	0.2898	0.2988	0.3082	0.3180
1.4922	0.1467	0.1536	0.1622	0.1724	0.1839
1.5708	0.0000	0.0000	0.0000	0.0000	0.0000

4. COMPUTATIONS

The results of the numerical evaluation of the efficiency for σ in the range from 0.01 to 0.10 are presented in Table 1. For any value of σ the maximum efficiency is attained at the same value of the inclination angle $\alpha = 3\pi/8$ and increases as the σ increases.

5. DISCUSSION AND CONCLUSIONS

1. An explicit solution of the problem of the streamlining of a partially penetrable obstacle analogous to the classical Riabouchinsky flow is obtained.
2. For small values of the cavitation number $\sigma < 0.01$ the model gives the estimate of the free flow turbine efficiency $\approx$30–35%. As in the classical situation [4], the solution is not applicable for modeling real flows for large values of the cavitation number.

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